

# UNMIXED $r$ -PARTITE GRAPHS

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**ABSTRACT.** Unmixed bipartite graphs have been characterized by Ravindra and Villarreal independently. Our aim in this paper is to characterize unmixed  $r$ -partite graphs under a certain condition, which is a generalization of Villarreal's theorem on bipartite graphs. Also we give some examples and counterexamples in relevance this subject.

## 1. Introduction

In the sequel, we use [4] as reference for terminology and notation on graph theory.

Let  $G$  be a simple finite graph with vertex set  $V(G)$  and edge set  $E(G)$ . A subset  $C$  of  $V(G)$  is said to be a vertex cover of  $G$  if every edge of  $G$ , is adjacent with some vertices in  $C$ . A vertex cover  $C$  is called minimal, if there is no proper subset of  $C$  which is a vertex cover. A graph is called unmixed, if all minimal vertex covers of  $G$  have the same number of elements. A subset  $H$  of  $V(G)$  is said to be independent, if  $G$  has not any edge  $\{x, y\}$  such that  $\{x, y\} \subseteq H$ . A maximal independent set of  $G$ , is an independent set  $I$  of  $G$ , such that for every  $H \supsetneq I$ ,  $H$  is not an independent set of  $G$ . Notice that  $C$  is a minimal vertex cover if and only if  $V(G) \setminus C$  is a maximal independent set. A graph  $G$  is called well-covered if all the maximal independent sets of  $G$  have the same cardinality. Therefore a graph is unmixed if and only if it is well-covered. The minimum cardinality of all minimal vertex covers of  $G$  is called the covering number of  $G$ , and the maximum cardinality of all maximal independent sets of  $G$  is called the independence number of  $G$ .

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For determining the independence number see [6]. For relation between unmixedness of a graph and other graph properties see [1, 5, 9, 12].

Well-covered graphs were introduced by Plummer. See [7] for a survey on well-covered graphs and properties of them. For an integer  $r \geq 2$ , a graph  $G$  is said to be  $r$ -partite, if  $V(G)$  can be partitioned into  $r$  disjoint parts such that for every  $\{x, y\} \in E(G)$ ,  $x$  and  $y$  do not lie in the same part. If  $r = 2, 3$ ,  $G$  is said to be bipartite and tripartite, respectively. Let  $G$  be an  $r$ -partite graph. For a vertex  $v \in V(G)$ , let  $N(v)$  be the set of all vertices  $u \in V(G)$  where  $\{u, v\}$  be an edge of  $G$ . Let  $G$  be a bipartite graph, and let  $e = \{u, v\}$  be an edge of  $G$ . Then  $G_e$  is the subgraph induced on  $N(u) \cup N(v)$ . If  $G$  is connected, the distance between  $x$  and  $y$  where  $x, y \in V(G)$ , denoted by  $d(x, y)$ , is the length of the shortest path between  $x$  and  $y$ . A set  $M \subseteq E(G)$  is said to be a matching of  $G$ , if for any two  $\{x, y\}, \{x', y'\} \in M$ ,  $\{x, y\} \cap \{x', y'\} = \emptyset$ . A matching  $M$  of  $G$  is called perfect if for every  $v \in V(G)$ , there exists an edge  $\{x, y\} \in M$  such that  $v \in \{x, y\}$ . A clique in  $G$  is a set  $Q$  of vertices such that for every  $x, y \in Q$ , if  $x \neq y$ ,  $x, y$  lie in an edge. An  $r$ -clique is a clique of size  $r$ .

Unmixed bipartite graphs have already been characterized by Ravindra and Villarreal in a combinatorial way independently [8, 11]. Also these graphs have been characterize in an algebraic method [10].

In 1977, Ravindra gave the following criteria for unmixedness of bipartite graphs.

**Theorem 1.1.** [8] *Let  $G$  be a connected bipartite graph. Then  $G$  is unmixed if and only if  $G$  contains a perfect matching  $F$  such that for every edge  $e = \{x, y\} \in F$ , the induced subgraph  $G_e$  is a complete bipartite graph.*

Villarreal in 2007, gave the following characterization of unmixed bipartite graphs.

**Theorem 1.2.** [11, Theorem 1.1] *Let  $G$  be a bipartite graph without isolated vertices. Then  $G$  is unmixed if and only if there is a bipartition  $V_1 = \{x_1, \dots, x_g\}, V_2 = \{y_1, \dots, y_g\}$  of  $G$  such that: (a)  $\{x_i, y_i\} \in E(G)$ , for all  $i$ , and (b) if  $\{x_i, y_j\}$  and  $\{x_j, y_k\}$  are in  $E(G)$ , and  $i, j, k$  are distinct, then  $\{x_i, y_k\} \in E(G)$ .*

H. Haghighi in [3] gives the following characterization of unmixed tripartite graphs under certain conditions.

**Theorem 1.3.** [3, Theorem 3.2] *Let  $G$  be a tripartite graph which satisfies the condition (\*). Then the graph  $G$  is unmixed if and only if the following conditions hold:*

- (1) *If  $\{u_i, x_q\}, \{v_j, y_q\}, \{w_k, z_q\} \in E(G)$ , where no two vertices of  $\{x_q, y_q, z_q\}$  lie in one of the tree parts of  $V(G)$  and  $i, j, k, q$  are distinct, then the set  $\{u_i, v_j, w_k\}$  contains an edge of  $G$ .*
- (2) *If  $\{r, x_q\}, \{s, y_q\}, \{t, z_q\}$  are edges of  $G$ , where  $r$  and  $s$  belong to one of the three parts of  $V(G)$  and  $t$  belongs to another part, then the set  $\{r, s, t\}$  contains an edge of  $G$  (here  $r$  and  $s$  may be equal).*

In the above theorem, he has considered the condition (\*) as:  
being a tripartite graph with partitions

$$U = \{u_1, \dots, u_n\}, V = \{v_1, \dots, v_n\}, W = \{w_1, \dots, w_n\},$$

in which  $\{u_i, v_i\}, \{u_i, w_i\}, \{v_i, w_i\} \in E(G)$ , for all  $i = 1, \dots, n$ .

Also to simplify the notations, he has used  $\{x_i, y_i, z_i\}$  and  $\{r_i, s_i, t_i\}$  as two permutations of  $\{u_i, v_i, w_i\}$ .

We give a characterization of unmixed  $r$ -partite graphs under certain condition which we name it (\*) (see Theorem 2.3).

In both theorems 2.1 and 2.2 in an unmixed connected bipartite graph, there is a perfect matching, with cardinality equal to the cardinality of a minimal vertex cover, i.e.  $\frac{|V(G)|}{2}$ . An unmixed graph with  $n$  vertices such that its independence number is  $\frac{n}{2}$ , is said to be very well-covered. The unmixed connected bipartite graphs are contained in the class of very well-covered graphs. A characterization of very well-covered graphs is given in [2].

## 2. A generalization

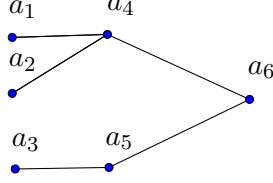
By the following proposition, bipartition in connected bipartite graphs is unique.

**Proposition 2.1.** *Let  $G$  be a connected bipartite graph with bipartition  $\{A, B\}$ , and let  $\{X, Y\}$  be any bipartition of  $G$ . Then  $\{A, B\} = \{X, Y\}$ .*

*Proof.* Let  $x \in A$  be an arbitrary vertex of  $G$ . Then  $x \in X$  or  $x \in Y$ . without loss of generality let  $x$  be in  $X$ . Let  $a \in A$ . then  $d(x, a)$  is even. Then  $a$  and  $x$  are in the same part (of partition  $\{X, Y\}$ ). Then  $A \subseteq X$ ,

and by the same argument we have  $X \subseteq A$ . Therefore  $A = X$ , and then  $\{A, B\} = \{X, Y\}$ .  $\square$

The above fact for bipartite graphs, is not true in case of tripartite graphs, as shown in the following example.



In the above graph there are two different tripartitions:

$$\{\{a_1, a_2, a_3\}, \{a_4, a_5\}, \{a_6\}\}$$

and

$$\{\{a_1, a_2\}, \{a_4, a_5\}, \{a_3, a_6\}\}.$$

A natural question refers to find criteria which characterize a special class of unmixed  $r$ -partite ( $r \geq 2$ ) graphs.

In the above two characterizations of bipartite graphs, having a perfect matching is essential in both proofs. This motivates us to impose the following condition.

*We say a graph  $G$  satisfies the condition  $(*)$  for an integer  $r \geq 2$ , if  $G$  can be partitioned to  $r$  parts  $V_i = \{x_{1i}, \dots, x_{ni}\}, (1 \leq i \leq r)$ , such that for all  $1 \leq j \leq n$ ,  $\{x_{j1}, \dots, x_{jr}\}$  is a clique.*

**Lemma 2.2.** *Let  $G$  be a graph which satisfies  $(*)$  for  $r \geq 2$ . If  $G$  is unmixed, then every minimal vertex cover of  $G$ , contains  $(r - 1)n$  vertices. Moreover the independence number of  $G$  is  $n = \frac{|V(G)|}{r}$*

*Proof.* Let  $C$  be a minimal vertex cover of  $G$ . Since for every  $1 \leq j \leq n$ , the vertices  $x_{j1}, \dots, x_{jr}$  are in a clique,  $C$  must contain at least  $r - 1$  vertices in  $\{x_{j1}, \dots, x_{jr}\}$ . Therefore  $C$  contains at least  $(r - 1)n$  vertices. By hypothesis  $\bigcup_{i=1}^{r-1} V_i$  is minimal vertex cover with  $(r - 1)n$  vertices, and  $G$  is unmixed. Then every minimal vertex cover of  $G$  contains exactly  $(r - 1)n$  elements. The last claim can be concluded from this fact that the complement of a minimal vertex cover, is an independent set.  $\square$

Now we are ready for the main theorem.

**Theorem 2.3.** *Let  $G$  be an  $r$ -partite graph which satisfies the condition  $(*)$  for  $r$ . Then  $G$  is unmixed if and only if the following condition hold: For every  $1 \leq q \leq n$ , if there is a set  $\{x_{k_1 s_1}, \dots, x_{k_r s_r}\}$  such that*

$$x_{k_1 s_1} \sim x_{q1}, \dots, x_{k_r s_r} \sim x_{qr},$$

*then the set  $\{x_{k_1 s_1}, \dots, x_{k_r s_r}\}$  is not independent.*

*Proof.* Let  $G$  be an arbitrary  $r$ -partite graph which satisfies the condition  $(*)$  for  $r$ .

Let  $G$  be unmixed. We prove that mentioned condition holds. Assume the contrary. Let

$$x_{k_1 s_1} \sim x_{q1}, \dots, x_{k_r s_r} \sim x_{qr},$$

but the set  $\{x_{k_1 s_1}, \dots, x_{k_r s_r}\}$  is independent. Then there is a maximal independent set  $M$ , such that  $M$  contains this set. Since  $M$  is maximal,  $C = V(G) \setminus M$  is a minimal vertex cover of  $G$ . Since the set  $\{x_{k_1 s_1}, \dots, x_{k_r s_r}\}$  is contained in  $M$ , then its elements are not in  $C$ , and since  $C$  is a cover of  $G$ , then all vertices  $x_{qi}$ , ( $1 \leq i \leq r$ ) are in  $C$ . But by Lemma 3.2, every minimal vertex cover, contains  $n - 1$  vertices of clique  $q$  th, a contradiction.

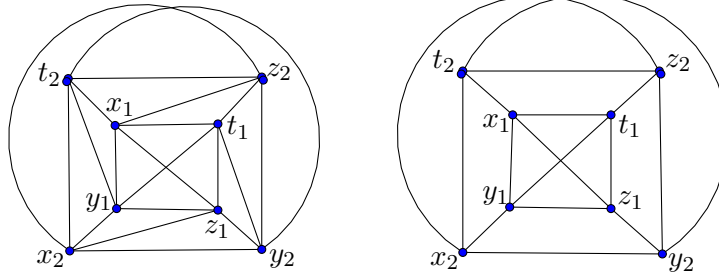
Conversely let the condition hold. We have to prove that  $G$  is unmixed. We show that all minimal vertex covers of  $G$ , intersect the set  $\{x_{q1}, \dots, x_{qr}\}$  in exactly  $r - 1$  elements (for every  $1 \leq q \leq n$ ). Let  $C$  be a minimal vertex cover and  $q$  be arbitrary. Since  $C$  is a vertex cover and  $\{x_{q1}, \dots, x_{qr}\}$  is a clique, then  $C$  intersects this set at least in  $r - 1$  elements. Let the contrary. Let the cardinality of  $C \cap \{x_{q1}, \dots, x_{qr}\}$  be  $r$ . Attending to minimality of  $C$ , for every  $1 \leq i \leq r$ ,  $N(x_{qi})$  contains at least one element, distinct from the elements of  $\{x_{q1}, \dots, x_{qr}\} \setminus \{x_{qi}\}$ , which is not in  $C$ , because we can not remove  $x_{qi}$  of cover. Let this element be  $x_{k_i s_i}$  where  $s_i \neq i$  and  $k_i \neq q$ . Then  $x_{k_i s_i} \notin C$  and  $\{x_{k_i s_i}, x_{qi}\}$  is in  $E(G)$ . There is at least two elements  $i$  and  $j$  such that  $1 \leq i < j \leq r$  and  $s_i \neq s_j$ , because  $x_{qi}$  can not choose its adjacent vertex from the part  $i$ . Therefore the set  $\{x_{k_1 s_1}, \dots, x_{k_r s_r}\}$  contain at least two elements. Then by hypothesis, at least two elements, say  $a, b$  of  $\{x_{k_1 s_1}, \dots, x_{k_r s_r}\}$  are adjacent by an edge. Now  $C$  is a cover but  $a, b$  are not in  $C$ , a contradiction.  $\square$

**Remark 2.4.** *Villareal's theorem (Theorem 1.2) for bipartite graphs, and Haghighi's theorem (Theorem 1.3) for tripartite graphs, are special cases of Theorem 2.3 (where  $r = 2$ , and  $r = 3$ ).*

### 3. Examples and counterexamples

In this section, we give examples of two classes of unmixed graphs, and an example which shows that it is not necessary that an unmixed  $r$ -partite graph satisfies condition (\*).

**Example 3.1.** *By Theorem 2.3, the following 4-partite graphs are unmixed.*



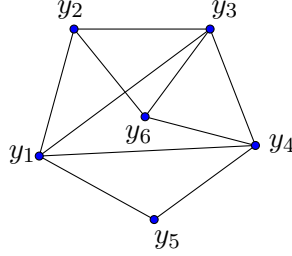
In each of the above graphs, there are two complete graphs of order 4 and some edges between them.

For  $r > 4$ , also  $r = 3$ , using two complete graphs of order  $r$ , we can construct  $r$ -partite unmixed graphs which are natural generalization of the above graphs.

**Example 3.2.** *For every  $n$ ,  $n \geq 3$ , the complete graph  $K_n$ , is an  $n$ -partite graph which satisfies the condition (\*). By Theorem 2.3,  $K_n$  is unmixed.*

Theorem 2.3 dose not characterize all unmixed  $r$ -partite graphs. More precisely, the condition (\*) is not valid for all unmixed graphs. In the following, we give an example of an unmixed  $r$ -partite graph which dose not satisfy the condition (\*).

**Example 3.3.** *The following graph is a 4-partite graph with partition  $\{y_1\}$ ,  $\{y_2, y_4\}$ ,  $\{y_3\}$ , and  $\{y_5, y_6\}$ . This graph dose not satisfy the condition (\*) because 6 is not a multiple of 4.*



We show that this graph is unmixed. Let  $C$  be an arbitrary minimal vertex cover of  $G$ . We show that  $C$  is of size 4.

Since  $C$  is a cover, it selects at least one element of  $\{y_4, y_6\}$ . Now we consider the following cases:

**case 1:**  $y_6 \in C$  and  $y_4 \notin C$ . In this case, since  $C$  is a vertex cover,  $y_1, y_3, y_5 \in C$ . Now  $\{y_1, y_3, y_5, y_6\}$  is a vertex cover of  $G$ , and since  $C$  is minimal,  $C = \{y_1, y_3, y_5, y_6\}$ .

**case 2:**  $y_4 \in C$  and  $y_6 \notin C$ . In this case,  $y_2, y_3 \in C$ , and at least one vertex of  $y_1, y_5$  and by minimality, only one is in  $C$ . Now since  $\{y_2, y_3, y_4, y_i\}$  where  $i \in \{1, 5\}$  is one of two vertices  $y_1$  and  $y_5$ , is a cover of  $G$ , by minimality of  $C$ ,  $C = \{y_2, y_3, y_4, y_i\}$ .

**case 3:**  $y_4, y_6 \in C$ . In this case, at least one of two vertices  $y_1, y_5$  and by minimality of  $C$ , only one is in  $C$ . Now if  $y_5 \in C$ ,  $y_3$  should be in  $C$  (because the edge  $\{y_1, y_3\}$  should be covered). Also  $y_2 \in C$  (because the edge  $\{y_1, y_2\}$  should be covered). Now  $\{y_2, y_3, y_5, y_4, y_6\}$  is a cover, and since  $C$  is minimal,  $C = \{y_2, y_3, y_5, y_4, y_6\}$ , that is a contradiction because  $y_6$  can be removed. If  $y_1 \in C$ , at least one of  $y_2$  and  $y_3$ , and by minimality only one, is in  $C$ . Now since  $\{y_1, y_4, y_6, y_j\}$ , where  $j \in \{2, 3\}$  is one of two vertices  $y_2$  and  $y_3$ , is a vertex cover, by minimality of  $C$ ,  $C = \{y_1, y_4, y_6, y_j\}$ .

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